MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 8

$j(\mathbb{M})/G := \{p \in j(\mathbb{M}) \mid \pi(p) \in G\}$, where $\pi : j(\mathbb{M}) \to \mathbb{M}$ is the projection.

Definition 1. *Let $\mathbb{Q}^* := \{q \in j(\mathbb{Q}) \mid q \restriction \kappa = \emptyset\}$. I.e. conditions are q with* $\operatorname{dom}(q) \subset j(\kappa) \setminus \kappa, |\operatorname{dom}(q)| < j(\kappa)$, *and for all* $\alpha \in \operatorname{dom}(q)$, $1_{Add(\omega,\alpha)} \Vdash q(\alpha) \in \dot{Add}(\omega_1, 1)$. *We have that $r \leq_{\mathbb{Q}^*} q$ iff*

(1) $\operatorname{dom}(r) \supset \operatorname{dom}(q)$;
(2) *for all* $\alpha \in \operatorname{dom}(q)$, $1_{Add(\omega,\alpha)} \Vdash r(\alpha) \leq_{\dot{Add}(\omega_1,1)} q(\alpha)$.

Lemma 2. *In $V[G]$, $j(\mathbb{M})/G$ is a projection of $\mathbb{P}^* \times \mathbb{Q}^*$.*

Proof. Suppose that $H \times K$ is a $\mathbb{P}^* \times \mathbb{Q}^*$-generic over $V[G]$. We have to show that in $V[G][H][K]$ there is a generic object for $j(\mathbb{M})/G$ over $V[G]$.

In $V[G]$ define $E = \{(p', q') \in j(\mathbb{M})/G \mid (\exists(p, q) \in j(\mathbb{M})/G)(p, q) \leq (p', q'), p \restriction j(\kappa) \setminus \kappa \in H, q \restriction j(\kappa) \setminus \kappa \in H\}$. We claim that E is $j(\mathbb{M})/G$-generic over $V[G]$. It is straightforward to check that this is a filter. For genericity, suppose that D is a dense subset of $j(\mathbb{M})/G$. Then let $D^* := \{(p, q) \in \mathbb{P}^* \times \mathbb{Q}^* \mid (\exists(p', q') \in D)(p' \restriction j(\kappa) \setminus \kappa = p, q' \restriction j(\kappa) \setminus \kappa = q)\}$ is a dense subset of $\mathbb{P}^* \times \mathbb{Q}^*$.

Let $(p, q) \in D^* \cap H \times K$. Let $(p', q') \in D$ witness that $(p, q) \in D$. But then by definition, $(p', q') \in E$.

□

Lemma 3. *In $V[G]$, $\mathbb{Q}^*$ is ω_1-closed, and $\mathbb{P}^*$ is ω_1-Knaster.*

Proof. Suppose that $\langle q_n \mid n < \omega\rangle$ is a decreasing sequence of conditions in $\mathbb{Q}^*$. We define a lower bound q, by setting $\operatorname{dom}(q) = \cup_n \operatorname{dom}(q_n)$. For $\alpha \in \operatorname{dom}(q)$, let $k < \omega$ be such that $\alpha \in \operatorname{dom}(q_k)$. Then for all $n \geq k$, $\alpha \in \operatorname{dom}(q_n)$. Moreover, since for all $k \geq n_1 < n_2$, we have that $1_{Add(\omega,\alpha)} \Vdash q_{n_2}(\alpha) \leq q_{n_1}(\alpha)$, we have that $1_{Add(\omega,\alpha)} \Vdash$ "$\langle q_n(\alpha) \mid n \geq k\rangle$ is a decreasing sequence in $\dot{Add}(\omega_1, 1)$". Therefore, there is some name σ, such that $1_{Add(\omega,\alpha)} \Vdash$ "$(\forall n \geq k)\sigma \leq_{\dot{Add}(\omega_1,1)} q_n(\alpha)$ [1]. Set $q(\alpha) = \sigma$. Then $q \leq_{\mathbb{Q}^*} q_n$ for all n, and so $\mathbb{Q}^*$ is ω_1-closed.

The second part of the lemma follows by a Δ-system argument. □

So, we know that T has an unbounded branch in $V[G][H][K]$. Next we will use some branch preservation lemmas to show that forcing with $\mathbb{P}^* \times \mathbb{Q}^*$ cannot add new branches, and so T must already have a branch in $V[G]$. We use the following lemma. The proof is left as an exercise.

[1] This is due to the fact that if $p \Vdash (\exists x)\phi(x)$, then there is a name a, such that $p \Vdash \phi(a)$.

Lemma 4. *(The product lemma) Suppose that $\mathbb{P}, \mathbb{Q}$ are two posets in a ground model V'. Suppose that H^* is $\mathbb{P} \times \mathbb{Q}$-generic over V'. Let $H = \{p \in \mathbb{P} \mid (\exists q \in \mathbb{Q})(p, q) \in H^*\}$ and $K = \{p \in \mathbb{Q} \mid (\exists p \in \mathbb{P})(p, q) \in H^*\}$. Then $V'[H^*] = V'[H][K] = V'[K][H]$.*

Conversely, if H is $\mathbb{P}$-generic over V' and K is $\mathbb{Q}$-generic over $V'[H]$, then H is $\mathbb{P}$-generic over $V'[K]$, and again $V'[H][K] = V'[K][H]$.

Then by the product lemma, $V[G][H][K] = V[G][K][H]$.

Proposition 5. *T has an unbounded branch in $V[G][K]$.*

Proof. In $V[G][K]$, T is a tree of height ω_1. Since $\mathbb{P}^*$ is ω_1-Knaster, it cannot add new branches. □

Proposition 6. *T has an unbounded branch in $V[G]$.*

Proof. In $V[G]$, T is an $\aleph_2$-tree, and $\mathbb{Q}^*$ is ω_1-closed. Moreover, $2^\omega = \omega_2$. So, by Silver's theorem $\mathbb{Q}^*$ cannot have added a new branch.

□

Corollary 7. *The tree property at $\aleph_2$ holds in $V[G]$.*

It turns out that the tree property at $\aleph_2$ is equiconsistent with the existence of a weak compact cardinal:

Theorem 8. *(Silver) Suppose in V, the tree property at $\aleph_2$ holds. Then in L, $\aleph_2^V$ is weakly compact.*

Below we summarize further results, motivated by Mitchell's theorem:

(1) (Abraham) Starting from a supercompact and a weakly compact, one can get the tree property simultaneously at $\aleph_2$ and $\aleph_3$.
(2) (Cummings and Foreman) Starting from ω many supercompacts, one can get the tree property simultaneously at $\aleph_n$ for all $2 \leq n < \omega$.
(3) (Neeman) Starting from ω many supercompacts, one can get the tree property simultaneously at $\aleph_n$ for all $2 \leq n < \omega$ and at $\aleph_{\omega+1}$.
(4) (Friedman-Halilovic /Gitik) From some (not too) large cardinals, one can get the tree property at $\aleph_{\omega+2}$, $\aleph_\omega$ strong limit.

What about combining $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$? The difficulty is that in order to get the tree property at $\aleph_{\omega+2}$ when $\aleph_\omega$ is strong limit, we have to have $2^{\aleph_\omega} > \aleph_{\omega+1}$, i.e. the negation of the *singular cardinal hypothesis* at $\aleph_\omega$. And constructions that do that tend to be fairly complicated. The following remains open:

(1) Is it consistent to have the tree property at $\aleph_{\omega+1}$ together with not SCH at $\aleph_{\omega+1}$? (For $\aleph_{\omega^2}$ the answer is yes.)
(2) Is it consistent to have the tree property simultaneously at κ^+ and κ^{++} when κ is strong limit singular?
(3) Is it consistent to have the tree property simultaneously at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ when $\aleph_\omega$ is strong limit?